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Phase jumps

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An earlier paper (Howe 1967) considered a non-linear theory of open-channel steady flow of deep water past a slowly modulated wavy wall. The wave pattern on the free surface of the water was obtained as the solution of a stably posed elliptic Cauchy problem, the main feature of the solution being the appearance of a 'shock' across which there is an abrupt change of phase. Such phase jumps can occur in a wide range of similar problems, but the advantage of the present case is that it is rather well suited to experimental investigation. This paper is therefore a lead-in to the more general problem of phase jumps, and uses the principle of conservation of energy in conjunction with the earlier solution to predict the possible position of the discontinuity on the free surface of the water. The possible nature of the free surface in the vicinity of the phase jump is also discussed (figure 4). This is a region where the width of the wave troughs becomes dramatically shorter than that of the neighbouring troughs. An approximate method of determining the line along which the phase jump occurs, not depending on a knowledge of the solution of the Cauchy problem, is also presented.

1. Introduction

In an earlier paper (Howe 1967) a non-linear theory of steady flow of deep water past a slowly modulated wavy wall was considered. The discussion was based on a theory proposed by Whitham (1965*a*, *b*) which describes the dispersion of slowly varying wave trains of large amplitude. Briefly, the method consists in first supposing the wave train to be locally a close approximation to an exactly periodic solution of the full non-linear equations of motion, from which an average Lagrangian is calculated in terms of the wave parameters. The dispersion equations governing the slow variation of these parameters are then obtained by an application of Hamilton's Principle.

Previous calculations of the dispersion of large amplitude wave-groups (Lighthill 1965, 1967; Whitham 1967*a*) had shown that in the particular case in which the dispersion equations formed an elliptic system a certain instability in the solutions of the initial value problems arose. Experimental confirmation of such instability, in the case of the development in time and space of a one-dimensional finite amplitude wave-group on deep water, has been reported by Benjamin & Feir (1967). However, because of the experimental difficulties involved in making accurate observations of moving wave trains, it appeared desirable to do a non-linear calculation for a case in which the wave pattern is steady. Such steady wave patterns could in principle be observed from a ship moving at constant velocity relative to still water but, because of difficulties mentioned by Howe (1967), it is not immediately clear how to apply the Whitham theory to the general ship-wave problem. The simpler problem of steady flow of deep water past a wavy wall of locally sinusoidal shape, whose amplitude decays slowly from a central maximum, was therefore considered, as a possible introduction to the ship-wave case. As explained in some detail in the earlier paper, the dispersion equation describing the wave-field on the free surface of the water is of elliptic type for moderate wall amplitudes, provided the wavelength of the wall is not too large.

The result of the earlier calculation is shown in figure 1, which is a map of the curves of constant phase on the free surface of the water (corresponding to the wave-crests and troughs). The x-axis is taken from left to right along the mean surface of the wall, the origin being at the point of maximum wall amplitude, and the positive direction of the y-axis is outwards from the wall. All distances and wavelengths are quoted with respect to U^2/g as the unit of length, where U is the undisturbed free stream speed which is in the positive x-direction. The amplitude of the wall is exaggerated by a factor of 8 in the figure. The wave pattern was obtained as the solution of an initial value problem, the wave-number $\boldsymbol{\kappa}$ and the phase θ being prescribed along the line $y = 1$.

It can be seen from the figure that there is a region of the (x, y)-plane within which the solution is indeterminate. This will be referred to as the 'gap region'. It is argued in the earlier paper that this is a manifestation of a shock-like phenomenon, across which there are discontinuous jumps in the wave-number and phase, the direction and spacing of the wave-crests changing abruptly through the shock. The shock, or 'phase jump', forms largely as a result of amplitude dispersion, whereby the larger amplitude waves have the larger phase velocity. The present paper aims to complete the solution of the problem by predicting the position of the discontinuity using the solution obtained and shown in figure 1 on either side of the shock. This is described in §3, and involves an application of the principle of conservation of energy. Out of all possible positions of the shock only one ensures that this is not violated.

Comparison with the analogous case in gas dynamics suggests the possibility of error in the solution which is being used behind (i.e. to the right of) the gap region. Such errors are third order effects in gas dynamics, and can often be neglected in the case of weak shocks; however, misleading results may be obtained in the strong shock case. Hence, the possible need for the solution behind the gap region to be modified, with possible consequential changes in shock position, must not be overlooked.

The present calculation assumes therefore that the shock is sufficiently weak for the method to be valid. Support for this view comes from the observations of Benjamin & Feir (1967), which indicate that the instability of finite amplitude waves does not result in turbulent dissipation of energy, but rather a redistribution of the energy in the spectrum of the wave-group.

The relevant conservation equations are derived in the next section. In §3 the predicted position of the shock is found, and the predicted amount of phase jump across the shock is used to estimate the difference in the number of wave-crests entering and leaving the shock. The possible form of the free surface in the neighbourhood of the phase jump is obtained by taking cross-sections of the free surface at right-angles to the shock. Observations on this would form a rather precise test of the theory. Finally, in §4 a crude physical argument is given which interestingly enough enables one to predict rather well the line along which the phase jump lies, although the method does not give its location on the line correctly.

2. Conservation equations and the dispersion equation

Let $\mathscr{L}(\boldsymbol{\kappa}, \omega)$ denote the average Lagrangian per unit horizontal area for finite amplitude waves on deep water, where $\boldsymbol{\kappa} = (l, m)$ is the wave-number and ω the time frequency. The system is assumed to be non-dissipative, since in an almost uniform wave train velocity gradients are smoothly varying functions of position and so the effect of viscosity may be ignored. In this case Lighthill (1967) has given an explicit expression for $\mathscr{L}$ approximately valid for all possible amplitudes, and this has been used in the computations of Howe (1967).

The wave-field is described in terms of a phase function $\theta(x_i, t)$ (where t is the time), which is smoothly varying and takes successive integral multiple values of 2π on successive wave-crests. In terms of θ the wave-number and frequency are given by

$$\kappa_i = \partial\theta/\partial x_i, \quad \omega = -\partial\theta/\partial t. \tag{2.1}$$

Hamilton's Principle,

$$\delta \iint \mathscr{L}(\kappa_i, \omega)\, d\mathbf{x}\, dt = 0, \tag{2.2}$$

then leads to the Euler equation

$$\frac{\partial}{\partial t}(\mathscr{L}_\omega) = \frac{\partial}{\partial x_i}(\mathscr{L}_{\kappa i}) \tag{2.3}$$

which, as pointed out by Whitham (1967*b*), is the conservation equation representing the balance between changes in the space-like adiabatic invariant $\mathscr{L}_{\kappa_i}$ and the time-like adiabatic invariant $\mathscr{L}_\omega$. Other conservation equations may be derived from (2.2) by applying Noether's Theorem (Bogoliubov & Shirkov 1957, p. 20). In particular the invariance of $\mathscr{L}$ with respect to arbitrary time and space translations lead respectively to the energy equation

$$\frac{\partial}{\partial t}(\omega\mathscr{L}_\omega - \mathscr{L}) - \frac{\partial}{\partial x_i}(\omega\mathscr{L}_{\kappa i}) = 0 \tag{2.4}$$

and the momentum equation,

$$\frac{\partial}{\partial t}(\kappa_j\mathscr{L}_\omega) - \frac{\partial}{\partial x_i}(\kappa_j\mathscr{L}_{\kappa i} - \mathscr{L}\delta_{ij}) = 0. \tag{2.5}$$

The definitions (2.1) give the two further conservation equations

$$\frac{\partial\omega}{\partial x_i} + \frac{\partial\kappa_i}{\partial t} = 0, \quad \operatorname{curl}\boldsymbol{\kappa} = 0. \tag{2.6}$$

The first of (2.6) represents the conservation of wave-crests, and the second says that wave-crests originate and terminate only on the boundaries of the wave-

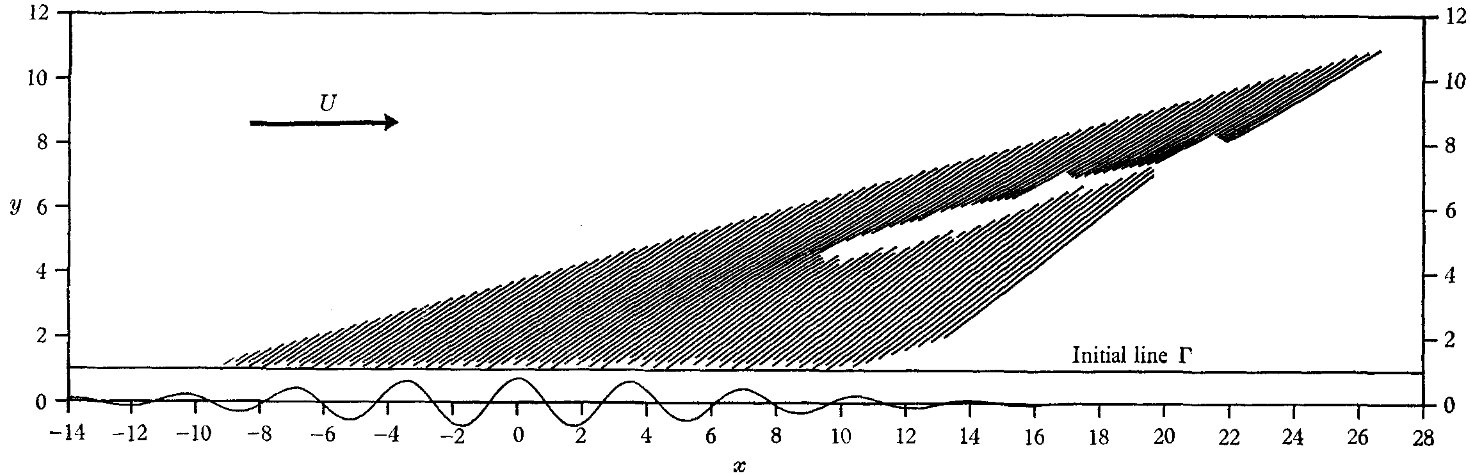

FIGURE 1. The surface wave pattern produced by steady flow at speed U past the wavy wall

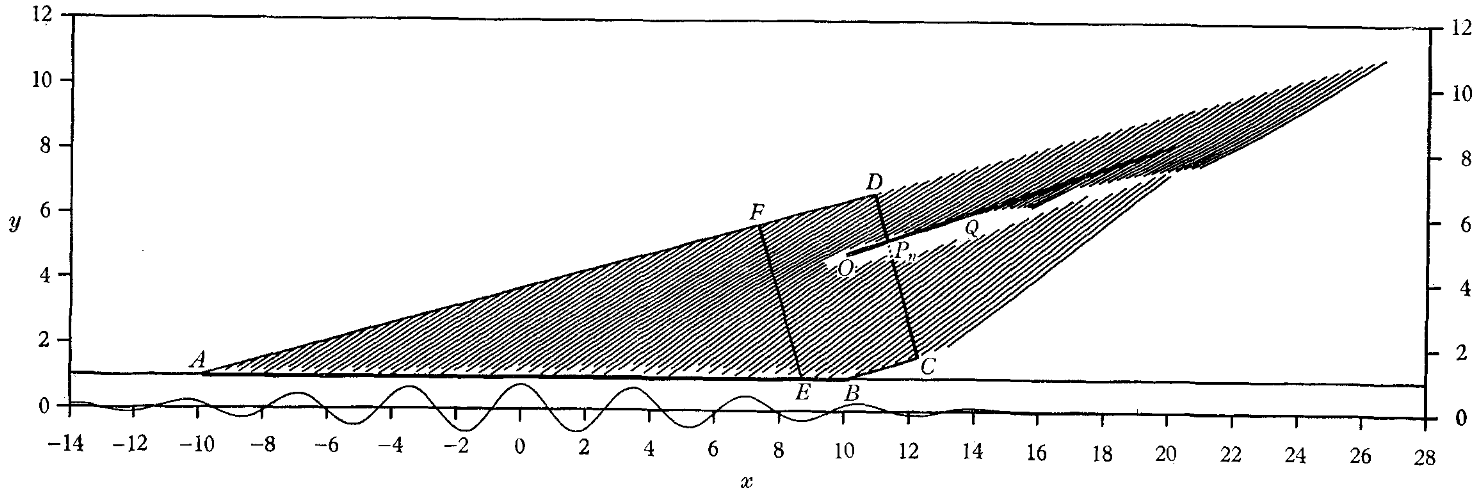

FIGURE 2. The position of the phase jump OQ as determined by the principle of conservation of energy. $ABCDA$ is a typical contour used to predict the P_n, and $AEFA$ is a typical test-integral contour.

field. However, (2.3) and (2.6) are valid only in those regions where the wave-field varies slowly on a scale of wavelength, whereas the energy and momentum equations (2.4), (2.5) are valid under all circumstances when the average Lagrangian $\mathscr{L}$ is replaced by the exact Lagrangian density.

The wavy wall problem concerns the steady-state distribution of waves on the free surface, and it is shown in the earlier paper how Lighthill's Lagrangian is expressed in terms of axes fixed relative to the wall, with respect to which the wave pattern is steady, so that in (2.1) and (2.6) $\omega = 0$ and $\partial/\partial t = 0$. The dispersion equation obtained from (2.1) and (2.3) becomes in this case

$$a(l,m)\,\theta_{xx} + 2b(l,m)\,\theta_{xy} + c(l,m)\,\theta_{yy} = 0, \tag{2.7}$$

where $$a(l,m) = \mathscr{L}_{ll}, \quad b(l,m) = \mathscr{L}_{lm}, \quad c(l,m) = \mathscr{L}_{mm}.$$

For the particular wall considered (2.7) is a quasi-linear elliptic equation, and the associated initial value problem was solved by the method of imaginary characteristics, the data l, m, θ being specified along the line $y = 1$.

3. The phase jump in the wavy wall problem

In this section the position of the phase jump in the wavy wall problem is predicted using the data obtained in the earlier paper on either side of the gap region. Discontinuous jumps across a shock are usually treated in terms of the physical conservation equations of the problem. Whitham (1967*b*) has pointed out that in non-linear problems of the present type there are always more conservation equations than the number of required shock conditions. Essentially one must distinguish between those conservation equations which remain valid in an un-averaged form, i.e. through the shock, and those which are true only for slowly varying wave trains. Thus, for example, it would be wrong to choose the wave conservation equations (2.6). However, the energy and momentum equations (2.4) and (2.5) may be used.

Consider in particular the energy conservation equation

$$\frac{\partial}{\partial t}(\omega\mathscr{L}_\omega - \mathscr{L}) - \frac{\partial}{\partial x_i}(\omega\mathscr{L}_{\kappa_i}) = 0. \tag{3.1}$$

Here $\mathscr{L} = \mathscr{L}(\kappa_i, \omega)$ is the Lighthill average Lagrangian, and is in terms of axes fixed relative to the mean motion of the water. In order that (3.1) may be written in terms of axes fixed relative to the wall, for which the wave pattern is steady, one first computes the derivatives $\mathscr{L}_\omega$ and $\mathscr{L}_{\kappa_i}$ in (3.1) and then replaces ω by $-Ul$ and $\partial/\partial t$ by $U_i(\partial/\partial x_i)$ where $U_i = (U, 0)$ is the undisturbed stream velocity, to obtain,

$$U_i\frac{\partial}{\partial x_i}[Ul\mathscr{L}_\omega + \mathscr{L}] - \frac{\partial}{\partial x_i}[Ul\,\mathscr{L}_{\kappa_i}] = 0. \tag{3.2}$$

Thus about any contour S in the (x, y)-plane

$$\oint_S [(Ul\mathscr{L}_\omega + \mathscr{L})U_i - Ul\,\mathscr{L}_{\kappa_i}]\,n_i\,ds = 0, \tag{3.3}$$

where n_i is the outward normal of S and ds the curvilinear length element. Equation (3.3) represents the balance of the energy fluxes into and out of S.

The position of the phase jump is determined by choosing a suitable sequence of contours S_n which pass through the gap region of the wavy wall solution. Since the solution is known on either side of this region one then extrapolates the integrand along the contour from either side of the gap region to a point P_n, say. At P_n there will be a discontinuous jump in the value of the integrand, and the precise location of P_n on the contour is adjusted to make the integral about the whole contour vanish. One may note that an integral relation of the form (3.3) can also be derived from the momentum equation (2.4). The same procedure applied in this case would lead to the same prediction of the position of P_n.

Figure 2 reproduces the solution of the wavy wall problem and shows also the position of the phase jump as determined by the above method. A typical integration contour is also shown, consisting of a segment AB from $x = -10$ to $+10$ of the initial line $y = 1$, a portion BC of the 'output curve' from the point $(10, 1)$ (see §4), a transversal CD cutting the shock approximately at right-angles, and the segment DA of the output curve from $(-10, 1)$. Before attempting to find the position of the jump, the reliability of the numerical solution obtained by Howe (1967) in satisfying (3.3) was examined by integrating around several closed contours not passing through the gap region, e.g., $AEFA$. The data available from the solution gave the wave-number $\boldsymbol{\kappa}$ to four decimal places, and when used in the test integrals the latter were found to differ from zero only in the fifth decimal place (the contributions along AE and EF separately were of order 1); in no case was the error in excess of $\pm 0{\cdot}00005$.

The sequence of contours S_n was generated by moving the component CD of the contour shown in figure 2 along AD and BC. The points P_n determined in this way were found to lie closely along a straight line, and the location of the phase jump in figure 2 is on a segment of the line

$$y = 0{\cdot}3372x + 1{\cdot}4875, \tag{3.4}$$

which is the least squares fit to the points P_n. The standard deviation of the fit is $0{\cdot}012$. The tip of the shock is rather ill-defined, but is approximately at the point O where $x = 9{\cdot}17$, and data for the line (3.4) extends as far as $x = 18{\cdot}61$.

The difference in the number of wave-crests which enter the segment OQ of the shock from the left (1) and which leave it on the right (2) is given by

$$\delta = \frac{1}{2\pi}\int_0^Q [\boldsymbol{\kappa}].d\mathbf{s}. \tag{3.5}$$

The integration is taken along the discontinuity, and the square bracket denotes the jump in wave-number. Since $\boldsymbol{\kappa} = \operatorname{grad}\theta$, and $\theta_1 = \theta_2$ at O,

$$\delta = \frac{1}{2\pi}\int_0^Q [d\theta] = \frac{1}{2\pi}(\theta_2 - \theta_1)_Q. \tag{3.6}$$

The variation of $2\pi\delta$ with distance s of Q along the jump from O is shown in figure 3, for possible future comparison with experiment.

One may illustrate the possible nature of the free surface of the water in the neighbourhood of the phase jump by plotting the surface elevation along a line such as CD in figure 2, cutting the jump at right angles. The free surface elevation is obtained from the solution of the earlier paper by substituting the amplitude and phase into the Stokes expansion for finite amplitude surface waves. Possible forms for the surface in the gap region may then be inferred by graphical interpolation between the curves of the known elevation on either side. By taking a large number of such cross-sections and plotting them on the same diagram a picture of the surface which seems almost three-dimensional is obtained.

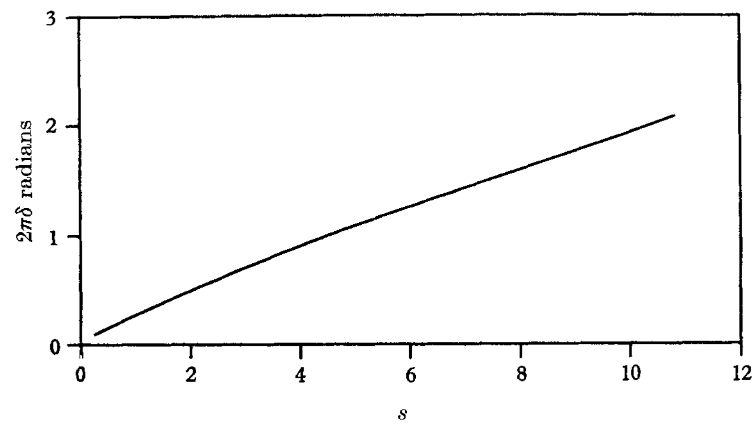

FIGURE 3. $2\pi\delta$ radians is the increase in phase in passing across the discontinuity in the positive x-direction at a distance s from the tip O of the phase jump.

This has been done in figure 4. Each wavy curve represents a cross-section of the free surface (magnified by a factor of 5) along a line such as CD in figure 2. Such a line is taken as the abscissa and the positive direction of the ordinate is in the sense OQ of figure 2. Thus figure 4 should be viewed by looking from the left along the direction of the phase jump line, which is also shown. With respect to this orientation the left-hand tip of each curve occurs where the wave energy is very small and so lies approximately at the undisturbed level of the free surface. The broken-line segments indicate a possible form for the free surface in the gap region.

The effect of the phase jump is rather dramatically emphasized by the shortening of the widths of those troughs shown entering the jump from below. Comment may also be made on the change in form of the crest which passes just to the left of the jump origin. The wave is of characteristic trochoidal section which becomes more peaked as it nears the tip of the jump. However, this tendency towards breaking is apparently arrested by the appearance of the jump which effectively acts as a means of transfer of wave energy from that crest, which moves on above the jump, to the waves in the region beneath the jump, so that the wave crest emerges on the other side much reduced in amplitude.

4. An approximate method for locating the line along which the jump lies

This discussion of the wavy wall problem is conveniently concluded by a consideration of an approximate method of predicting the line along which the phase jump is formed, which gives rather more physical insight than the purely numerical approach.

Consider the dispersion equation (2.7)

$$a\theta_{xx} + 2b\theta_{xy} + c\theta_{yy} = 0.$$

The characteristics of this equation are given by

$$ady^2 - 2bdxdy + cdx^2 = 0,$$

or, setting $\Delta = \sqrt{(ac-b^2)}$, and noting that the equation is elliptic, by

$$\frac{dx}{dy} = \frac{b}{c} \pm i\frac{\Delta}{c}. \tag{4.1}$$

Remembering that the dispersion equation was originally solved by analytic continuation into the complex x-plane, (4.1) states that, if y is real, the solution at any point P, say, in the real (x, y)-plane is determined by the two conjugate characteristics through P originating at two conjugate points in that complex x-plane which meets the real (x, y)-plane along $y = 1$. Data at these points is obtained by keeping the real part of x fixed and analytically continuing the data from the initial line Γ (i.e. $y = 1$) into the complex x-plane. At P these two characteristics define a plane element cutting the real (x, y)-plane along the curve

$$dx/dy = b/c. \tag{4.2}$$

As a first approximation (4.2) may be regarded as defining the rays along which energy is propagated. Indeed in the case of infinitesimal amplitude they *do* degenerate into the steady-state analogue of the group velocity lines, since

$$\frac{dx}{dy} = \frac{b}{c} = \frac{\mathscr{L}_{lm}}{\mathscr{L}_{mm}} = \frac{\partial(\mathscr{L}_m, m)\,.\,\partial(m, l)}{\partial(l, m)\,.\,\partial(\mathscr{L}_m, l)} = -\left(\frac{\partial m}{\partial l}\right)_{\mathscr{L}_m}.$$

For finite amplitudes the neglect of the imaginary part of (4.1) implies that the propagation of changes along the approximate 'rays' (4.2) would be to some extent blurred.

Suppose now that the wall has constant amplitude. The solution of the dispersion equation is then trivially a plane wave, and the energy rays (4.2) become the family of parallel straight lines

$$x = \frac{b}{c}y + \text{const.} \tag{4.3}$$

Here b/c is a function of κ which is constant. But when the amplitude of the wall varies, κ will vary along Γ. Provided that this variation is slow enough, in the sense of Whitham's approximation, the energy rays would be expected to remain straight for some distance from Γ, but the slope b/c would vary in accordance with the variation of κ along the initial line. It is conceivable that under these circumstances the family (4.3) would envelope a caustic along which the solution of the dispersion equation would be indeterminate. Physically one then expects to find a shock in the solution.

These ideas may be illustrated by referring to the results of the wavy wall calculation. The dispersion equation was solved by reducing it to a set of five

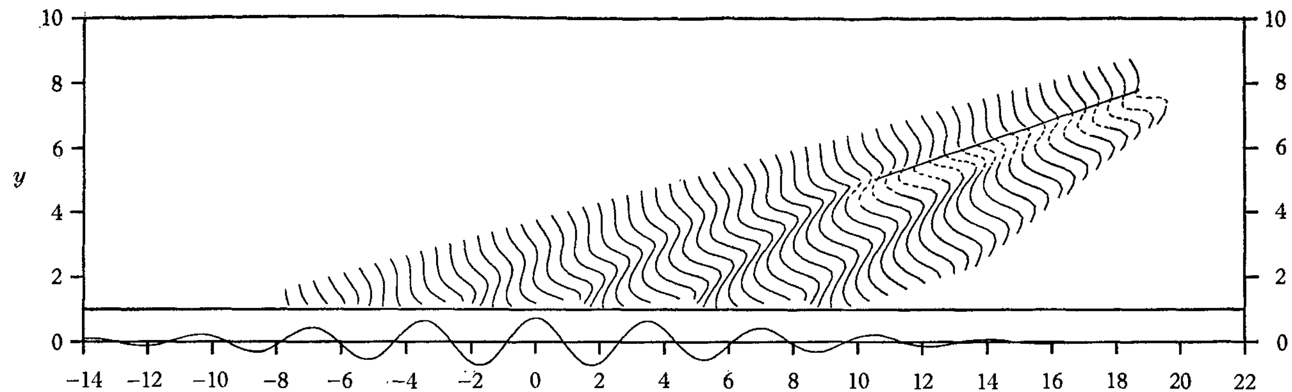

FIGURE 4. Each wavy curve is a cross-section of the free surface along a direction perpendicular to the phase jump. The figure should be viewed by looking from the left along the direction of the discontinuity. The broken-line segments indicate a possible form for the free surface in the neighbourhood of the phase jump.

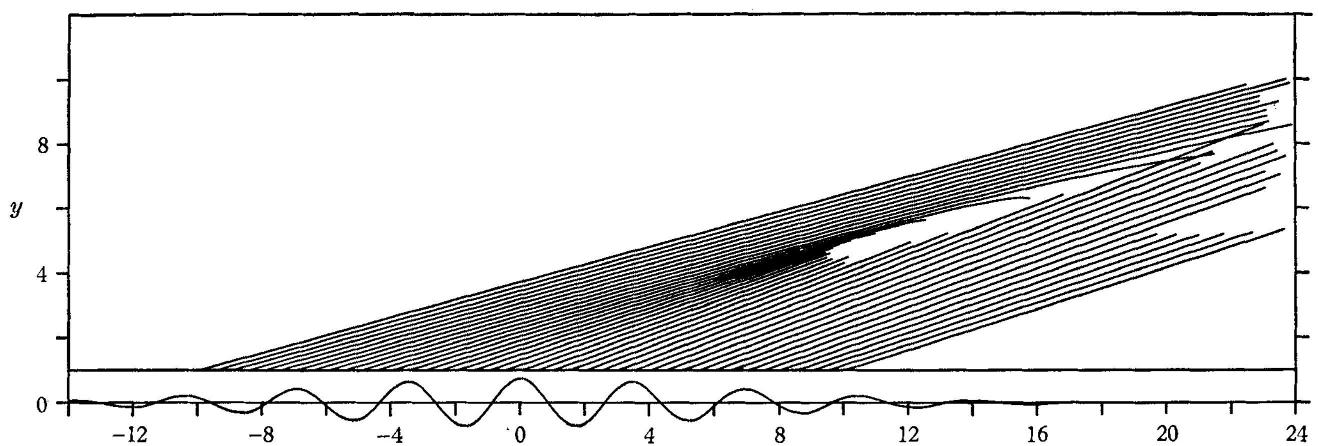

FIGURE 5. The 'output' curves obtained by Howe (1967). The method of imaginary characteristics used in solving the dispersion equation gives the values of l, m and θ along these rays.

first order partial differential equations in which the unknowns (x, y, l, m, θ) were to be determined in terms of co-ordinates (ξ, η) defined by

$$\xi = \frac{\alpha+\beta}{2}, \quad \eta = \frac{\alpha-\beta}{2i},$$

where α, β are the complex characteristic co-ordinates of the dispersion equation. The method involves continuing the initial data into the complex η-plane, where if $\eta = \lambda + i\sigma$, λ is held fixed at λ_0, say. In the (ξ, σ)-plane the system of equations is hyperbolic and is soluble by the method of characteristics. At $\sigma = 0$ the solution so obtained reduces to the solution in the real (ξ, η)-plane along the straight line $\eta = \lambda_0$. A complete covering of any portion of the real (ξ, η)-plane is obtained by repeating this procedure over a complete interval of λ_0.

Since the solution of the system of equations corresponds to a constant real value of η, it follows that the image curve in the (x, y)-plane of the straight line $\eta = \lambda_0$ satisfies

$$d\alpha - d\beta = 0, \tag{4.4}$$

where, as described in the earlier paper,

$$x = \lambda_0, \quad y = 1, \quad \text{on} \quad \alpha + \beta = 0. \tag{4.5}$$

This set of 'output curves' obtained in the wavy wall calculation is shown in figure 5. They are straight except near the region where breakdown occurs. Because of this it is valid to approximate to each by giving to $\boldsymbol{\kappa}$ the value it has where it cuts the boundary $y = 1$, and to solve (4.4) on the assumption that $\boldsymbol{\kappa}$ is constant. This approximation readily yields the solution

$$x = (b/c)y + \text{const.}, \tag{4.6}$$

the constant being determined by the starting-point on $y = 1$. Thus under these circumstances the output curves correspond to the energy rays (4.3).

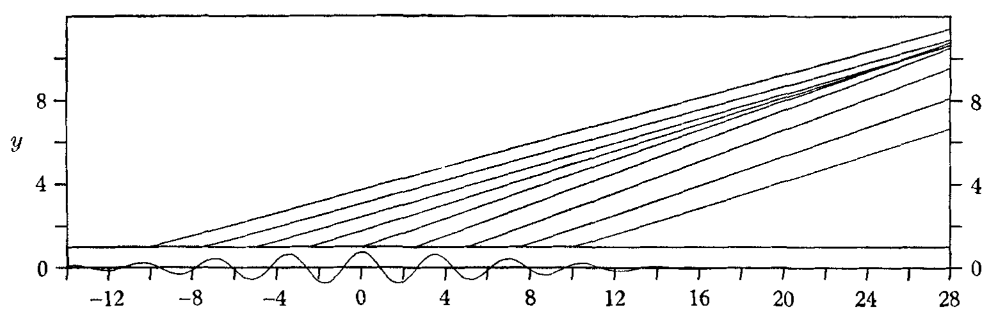

FIGURE 6. The family of 'energy rays' given by equation (4.3). The effect of finite amplitude is to blur the propagation of changes along these rays.

Figure 5 reveals that the output curves of the wavy wall calculation *do* in fact tend to form an envelope along the front of the gap region. This may be compared with figure 6 in which are drawn the energy rays (4.3) originating from the same points on $y = 1$ as the output curves. The energy rays form a cusp at (23·37, 9·21). This is quite a lot farther from the initial line than the start of the envelope in figure 5.

However, the line (3.4) along which the phase jump occurs is rather well determined by that along which the cusp forms in the case of the energy rays. Taking our definition of the latter line as that ray which touches the cusp at its tip, one obtains for its equation,

$$y = 0{\cdot}3235x + 1{\cdot}6609. \tag{4.7}$$

This is negligibly different from (3.4) within the range $x = 10, 20$ of validity of the phase-jump calculation. Thus the blurring of the energy rays may alter the distance to formation of the discontinuity but apparently does not alter the direction in which the phase-jump is formed.

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